Naṣīr al-Dīn al-Ṭūsī’s Version of The Measurement of the Circle of Archimedes from his Revision of the Middle Books
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Abstract
The mathematical treatises known as Tahrîr al-mutawassîtât, “Revision of the Middle books”, are a collection of Greek and Arabic treatises which were re-edited by Naṣīr al-Dīn al-Ṭūsî. This paper explores the nature of the editorial work Ṭūsî did, showing that it is not a simple rendering of the original text but that it involves major revisions and in fact original additions. For the first time, a collated version of Ṭūsî’s revision on the treatise on the measurement of the circle (by Archimedes) is provided. A translation of the Arabic text is put side by side with a rendition of the treatise as it was circulated before Ṭūsî’s time. An in-depth analysis of the text elucidates the nature of Ṭūsî’s contributions to the text. This article also provides modest corrections to earlier studies by W. R. Knorr and P. Luckey.

Keywords: Naṣīr al-Dīn al-Ṭūsî, Middle books, quadrature of circles, Archimedes, Measurement of the circle
Introduction

Next to Naṣīr al-Dīn al-Ṭūsī’s (d. 1274) impressive contributions to science in the fields of philosophy and astronomy, stand his equally notable contributions to mathematics. The collection called Revision of the ‘Middle books’ (Taḥrīr al-mutawassīṭāt) is such a contribution, being a collection of Arabic versions of mostly Greek treatises and to a lesser extent treatises of Arabic origin. Ṭūsī reorders propositions in a more logical way, resolves defective translations, provides comments, and introduces new proofs. The large collection of extant manuscripts shows in itself that it had a significant influence on mathematicians in the Islamic world in the centuries to come (see Sezgin, p. 131; Brockelmann, vol. I, pp. 510-511; Rosenfeld, pp. 212-213. Compare with Kheirandish, pp. 131-144). Influence on Western Europe is known to exist as well, as for example Gerard of Cremona’s Latin translation of The measurement of the Circle also indicates Ṭūsī’s additions (Clagett, pp. 40-55).

This article is meant foremost to show that indeed Ṭūsī did all of the aforementioned things, and next also show the kind of interventions he made on the original text. For this aim, the small treatise on the measurement of the circle by Archimedes will suffice. We will first provide some introductory comments and then provide the text of Ṭūsī’s revision of The measurement of the Circle, together with a translation which is compared with a rendition of Archimedes’ treatise as it was circulated before Ṭūsī’s time. This comparative presentation will virtually instantly reveal a lot of the nature of Ṭūsī’s editorial interventions and his additions. Afterwards we will elucidate the mathematics that are used and discuss some main characteristics of Ṭūsī’s revisions.

1. The research for this paper was partly conducted as a B.Sc. thesis at the Department of Mathematics, Utrecht University. I would like to thank Prof. J. P. Hogendijk and the anonymous reviewer for their valuable comments on previous versions of this paper.
Preliminary notes
Perhaps it is good to first briefly go over some background information, which will help us contextualize Ṭūsī’s revisions of the ‘Middle books’ in general, and of The measurement of the circle specifically. The term ‘Middle books’, Mutawassiṭāt, refers to a collection of Arabic translations of Greek mathematical treatises. Little is known about this term, the best study of it dating from 1865 by Moritz Steinschneider (see pp. 54-97). From this study we know that the term was used as early as the 10th century C.E. (Ibid, p. 56) In the centuries after that, the term was used as if the collection were one book in contrast to Euclid’s Elements and Ptolemy’s Almagest, which would point to an explanation of the term mutawassiṭāt as being the required study material after the Elements and before the Almagest (Ibid, pp. 56-58). This is confirmed by Nāṣīrī (fl. 11th c.) who states at the beginning of his rendering of (pseudo-) Archimedes’ Lemmas;

“… the Middle books […] which it is necessary to read between the book of Euclid and the Almagest.” (Ibid, p. 78)

As it became used as a didactical term over the years, so it also included treatises by Islamic authors rather than only Arabic translations of Greek treatises.

The measurement of the circle was written by Archimedes (ca. 250 B.C.E.) and underwent subsequently a remarkable history as a text. Over time, different versions began to circulate, supplanting previous ones. The original we have no more; what we nowadays consider as Archimedes’ treatise stems from Late Antiquity (Heiberg, pp. 232-234), at the latest from before the 6th century, since Eutocius’ famous commentary relies on a version close to it (Knorr, esp. p. 405 and p. 431). This treatise consists in its extant form of three propositions. The first one explains that a circle is equal to a right triangle of which one of the sides is equal to the radius and the other equal to the perimeter. The second proposition (in Ṭūsī’s text the third) relies on the third proposition and gives 22 over 7 as an approximation of π. The third proposition (in Ṭūsī’s text the second) gives for the first time in history a very good approximation of π (πi). The treatise as a whole is notoriously short. On several occasions the text merely states the
result and leaves it to the reader to check the reasoning.¹ We do not know much about the reception into the Islamic world; though it seems that the Arabic translation was made from an older version than the Greek text we have now (Knorr, p. 431). We at least know that The measurement of the circle was available before 857 C.E., as Kindī supposed to have corresponded on its content with the scientist Ibn Māsawayh (who died in 242/857; see Rashed, p. 15). The details of the original translation remain uncertain, although the text was most probably translated first by Isḥāq ibn Ḥunayn and afterwards retranslated or improved by Thābit ibn Qurrah (Sezgin, pp. 128-129). In his study of Kindī’s treatise, Rashed suspects there might have been another translation by Qustā ibn Lūqā (Rashed, pp. 15-16; see Lorch, pp. 94-114, who warns for a too ready acceptance of this). The commentary by Eutocius was also translated into Arabic. As we will see, it was not used by Ṭūsī to write his revision. Perhaps he did not have access to it.²

Ṭūsī wrote his revision of The measurement of the circle at around 661/1262-63. It appears that he wrote the revision of Menelaus’ Spherics later (663/1265; see Krause). Nevertheless, The measurement of the circle is the last treatise in both the print and facsimile used for this study and thus the actual compilation of the Taḥrīr al-mutawassīṭāt was either conducted by Ṭūsī at the very end of his life or, more probable, after he had died. At the beginning of his revision of On the sphere and cylinder by Archimedes Ṭūsī states that he read it first from a poor copy of Thābit ibn Qurrah and later in an old codex of Isḥāq ibn Ḥunayn. He then states:

“…In that codex I found what I had been looking for, and I thought of editing the book correctly, setting out its contents precisely, explaining its postulates by means of geometrical principles, adding its necessary premises, and providing a commentary on what is problematic about it, based on what I could take from the commentary of Eutocius or other books of

¹. Heath, for example, writes: “How did Archimedes arrive at these particular approximations? No puzzle has exercised more fascination upon writers interested in the history of mathematics.”, p. 51.
². Knorr assumes that he did consult Eutocius’ commentary, but does not present convincing evidence, cf. Knorr, p. 549.
this discipline [...] I added at the end of the book the treatise of Archimedes on the measurement of the circle, since this depends on postulates which are brought up in the former treatise.” (At-Tūsī, pp. 2-3)

Tūsī here not only provides us with a short description of his methodology, but also proposes that *The measurement of the circle* should be seen as an appendix to *On the sphere and cylinder*. Indeed, in the print and facsimile *The measurement of the circle* comes directly after *On the sphere and cylinder*. On the other hand, the layout of *The measurement of the circle* gives the impression of being a text on its own. The commentary of *On the sphere and cylinder* is properly concluded, and the new text is properly introduced (and likewise properly concluded).¹

**Formal remarks**

For the text and translation of Tūsī’s revision of *The measurement of the circle* a printed edition was used. In this edition it functions as an appendix of the larger ‘Book on the sphere and cylinder with revisions’ (*Kitāb al-kura wa-l-usṭūwāna li-Arkhimīdis bi-ta[t]rīr*), other texts in this book include revisions on Autolycos, Aristarchus, Hypsicles and another text of Archimedes. Unfortunately, this print cannot be trusted on its own, especially when it comes to the numbers written in *Abjad* system. For a better understanding of the text and to correct the Hyderabad printed edition, I collated it with a facsimile published in Tehran (H= Hyderabad print, T= Tehran facsimile).²

For the rendition of the Arabic translation of Archimedes’ text, as it must have circulated in Late Antiquity, I have made use of the facsimile of MS Fatih 3134, f. 2v-6v, as printed in Knorr’s study on the textual tradition of Archimedes’ text (Knorr, pp. 455-463). Since its use here is primarily to compare the mathematical content, a translation suffices. It should be pointed out that the Arabic translation is very close to the Greek text we have today, only departing from it in

¹. Though it may be pointed out that whereas *On the sphere and cylinder* does start with a *bismillah* (in God’s name), *The measurement of the circle* does not.

². al-Tusi, Nasir al-Din, *Tañbīr-e Mutawasītāt*, Introduction by Dr. Jafar Aghayanī-Chavoshi, Institute for Humanities and Cultural Studies, Tehran, 2005. This is a facsimile of Ms. Tabriz, Melli Library, no. 3484.
a significant way in the first proposition. Knorr supplies an English translation of both the Arabic translation and Ţūsī’s text, but the quality of this is disputable and thus I have here provided my own, though still drawing from Knorr, in particular for the rendition of the Arabic translation. The order of the propositions in the Arabic translation is the same as in the extant Greek text, but I have rearranged it here as to align it with Ţūsī’s text. The following table shows the transliteration of the labels in the geometrical figures.

### Usage of labels

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Text and translation of *Taksīr al-dāʿirah*

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<th><strong>Rendition of Arabic translation</strong>*</th>
<th><strong>Translation of Šūsī’s text</strong>*</th>
<th><strong>Arabic text of Naṣīr al-Dīn al-Ṭūsī</strong>*</th>
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<tr>
<td>Book of Archimedes on the measurement of the circle</td>
<td>Essay of Archimedes on the measurement of the circle, consisting of three propositions.</td>
<td>مقالة ارشمیدس فی تکسر الدائرہ یعنی سه اشكال الدائرہ فی مساوی نصف قطر الدائرہ يكون واحد ضلعه الخیطین بالزاوية المتساوية مساوی نصف قطر الدائرہ والاثانی مساویاً لمحیطها واصلها بتساوی سطح نصف قطرها فی الخط المسائی لصف محیطها</td>
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<tr>
<td><strong>1) Any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the base is equal to the circumference.</strong></td>
<td>Every circle is equivalent to a right triangle of which one of the two sides which surround the right angle is equal to the half of the diameter of that circle, and the other [side] is equal to its perimeter. The result is that it is equal to the rectangle contained by half of its diameter and the line that is equal to half of its perimeter. So let the circle be a circle <strong>abcd</strong> and the said triangle a triangle <strong>e</strong>. If the circle is not equal to it, then it is either bigger or smaller.</td>
<td>فلتنک الدائرہ دائرہ <strong>ابجد</strong> والمنت المذكور مثلث و فان لم تنک الدائرہ مساویاً له فیی إما أعظم منه و إما أصغر</td>
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Let it be bigger first. We make in the circle the square $ag$. Then from the circle $abgd$ a part bigger than its half has been separated, which is the square $ag$. We bisect the arc $afb$ and arcs similar to it at point $f$ and points similar to it, and we connect $af$, $fb$, and similarly for the others. Then from the remainder of the segments of the circle $a bgd$ again more than their half is removed, namely $afb$ and those similar to it. If we have done like that according to what follows, there will remain sections which are smaller than the measure of the excess of the circle over the triangle $e$.

Let it be bigger first. We draw in the circle a square $ag$. It separates from it a square bigger than the half of it [i.e. the circle]. We bisect $ab$ at $f$ and similarly the four arcs and we connect the chords. So the resulting triangles separate more than half of the sections as has been proved before.

This is repeated until there remains sections of the circle that are smaller than the measure of the excess of the circle over the triangle $e$.

The rectilinear figure that is contained in the circle is thus bigger than the triangle.

We make $n$ be the center and extend the perpendicular $ns$.

The line $ns$ is then less than one of the two sides of the triangle containing the right angle. The perimeter of the rectilineal figure is less than the other side of the two, because it is

\[\text{Perimeter of the circle} > \text{Perimeter of the triangle}\]

\[\text{The circle is thus bigger than the triangle.}\]

So the equilateral figure that is in the circle is [then] bigger than the triangle. Let the center be $n$ and we draw from there to one of the sides a perpendicular line, let it be $ns$. Then it is less than the line $nx$ which is equal to one of the sides of the triangle $e$ [i.e. the radius]. The perimeter of the equilateral figure is less than the perimeter of the circle [which] is

\[\text{The line is less than one of the two sides of the triangle.}\]

\[\text{The perimeter is less than the perimeter of the circle.}\]
less than the perimeter of the circle. The product of one of the sides of the triangle containing the right angle with the other, which is double the area of the triangle, is bigger than the product of \( ns \) with the perimeter of the polygon, which is double the area of the polygon. The same holds for half of that. Thus, the triangle is bigger than the polygon, while earlier it was smaller. This is no doubt a contradiction.

Let the circle be, if possible, less than the triangle \( e \), and let the square be circumscribed, which is \( oq \). Then more than half of the square \( oq \) has been separated, because of the circle. We divide the arc \( ba \) in two halves at \( f \), similarly we divide the other arcs in two halves, two halves. Let points of the sections be passed by lines perpendicular to the circle, so that line \( zT \) has been divided into two halves at point \( f \). Then line \( nq \) is perpendicular to \( zT \), the other lines being similarly.

Because \( qz \) and \( qT \) are equal to the other side of the triangle \( e \). So the rectangle contained by \( ns \) times the perimeter of the figure, I mean, twice the magnitude of the figure is less than twice the triangle. So the figure is less than the triangle and it is greater than it.

This is a contradiction.

Then let the circle be smaller than the triangle and we draw around it a square \( oq \). [The circle] separates from the square more than half of it.

We bisect the arc \( ba \) at \( f \) and draw \( zIT \), tangent to the circle at \( f \). The radius \( nf \) is perpendicular to it [i.e. the tangent line]. We do this likewise with the other arcs.

Because \( qb \) and \( qa \) are equal to the other side of the triangle \( e \). So the rectangle contained by \( ns \) times the perimeter of the figure, I mean, twice the magnitude of the figure is less than twice the triangle. So the figure is less than the triangle and it is greater than it.

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We bisect the arc \( ba \) at \( f \) and draw \( zIT \), tangent to the circle at \( f \). The radius \( nf \) is perpendicular to it [i.e. the tangent line]. We do this likewise with the other arcs.

Because \( qb \) and \( qa \) are
bigger than $T_z$, their halves are bigger than its half, so that line $qT$ is bigger than $T_T$ which is like $T_b$.

Thus, the triangle $qfT$ is bigger than half of the triangle $qfb$, so all the more it is bigger than half of the figure $qfyb$ which is contained by the line $bq$, $qf$, and the arc $bvf$. Similarly, the triangle $qfz$ is bigger than $fSaz$.

All of $Tqr$ is bigger than half of the figure $aSfybq$, and similarly it is for the other triangles, similarly even more so for the half of the other segments.

If we have done like that according to what follows, there will remain segments outside of the circle that, when combined, constitute less than the excess of the triangle $e$ over the circle $abgd$.

The four triangles at the angular points of the square separate a bigger [part] than half of the remainder of the square after subtraction of the circle. We divide the arcs exactly so, repeatedly, and we draw tangent lines to the circle until the remaining segments outside the circle together are smaller than the excess of the triangle $e$ over the circle. So the polygon that lies on the circle is smaller than the triangle $e$. 

[$\text{ jap }$] طب طف رز الاربعه متساوية يكون طف رز متساويين وهم معاً اطول من طف نز اطول من

[$\text{jbap }$] Filmt $qf$ اعظم من مثلث $bT$ الذي هو اعظم من قطعة $Tf$ الخارجه من الدائرة وكذلك في الباقى

[$\text{ japa }$] وإلین $qf$ اعظم من مثلث $fSaz$.

[$\text{ japi }$] وما المتتلتین $qf$ الاربعه التي على زوايا المربع تفصل من باقي المربع بعد تقسيم الدائرة هن اعظم من النصف ونصف التي هكذا مرة بعد أخرى وتخرج الخطوط المستقيمة للدائرة التي ان تبقى قطع خارجة من الدائرة مجموعها اصغر من زيادة مثلث $e$ على الدائرة فيكون الشكل الكثير الاضلاع الذي على الدائرة اصغر من مثلث $e$.
So let there remain the segment $fza$ and similar segments, then the rectilinear figure which contains the circle is smaller than the triangle $e$. [But] this is not possible because it was bigger than it. That is because $na$ is equal to the perpendicular of the triangle, while the perimeter of the polygon is bigger than the other side of the triangle which contains the right angle, because it is bigger than the perimeter of the circle, and the product of $fn$ into the perimeter of the polygon is bigger than the product of one of the two sides of the triangle containing the right angle into the other. Thus, the circle is not smaller than the triangle $e$, while it has previously been proved that it is not bigger than it.

Therefore the circle $abcd$ is equal to the triangle $e$.

Also, since the area of the triangle $e$ is equal to the product of its perpendicular into half its base, and its perpendicular is equal to half the diameter of the circle $abcd$ and its base.

But the rectangle contained by $nf$, the radius, and the perimeter of the figure that circumscribes the circle, I mean, twice the magnitude of the figure is greater than twice the triangle, because of the fact that the perimeter of the figure is greater than the perimeter of the circle. The figure is bigger than the triangle but it was smaller than it. That is a contradiction.

So the circle is equal to the triangle $e$, so the rectangle contained by the radius and half of the perimeter [of the circle] is equal to the surface of the circle and that is what we wanted.

فاذاً الدائرة مساوية بمثلث $e$، فسطح
نصف النفر في
نصف المحيط مساو لسطح الدائرة
وذلك ما اردناه
is equal to the circumference of the circle $abgd$, then the product of half the diameter into half the circumference of the circle $abg$ is equal to the area of the triangle $e$, and that is what we wanted to prove.

For this reason it is that the product of half the diameter into half the section of the circumference is the area of the figure which is contained by that section and the two lines which proceed from the two ends of the section to the center.

In addition it has been demonstrated with this [proof] that the rectangle contained by the radius and half of a part of the perimeter is equal to the circular sector that is contained by that part and the two lines by that part and the two lines issuing from the center to the two endpoints of that part.

2) Every circumference of a circle is greater than three times the diameter by less than a seventh of the diameter but greater than $\frac{10}{71}$ of the diameter.

Let $ag$ be the diameter of a circle, and its center $e$, the line $dz$ be tangent to the circle, the angle $zeg$ be a third of a right angle.

So the ratio $ez$ to $zg$ is $\frac{\sqrt{2}}{\sqrt{3}}$.
as the ratio 306:153, while the ratio of $eg$ to $zg$ is bigger than the ratio 265:153. If we subtract the square of the number that is opposite to $zg$ from the square of the number that is opposite to $ez$, and we take the root of what is left, $eg$ is in this magnitude more than 265 and the difference is some fraction [i.e. is less than one].

Let us divide angle $zeg$ in halves by the line $eh$, so that the ratio of $ze$ to $eg$ is as the ratio of $zh$ to $hg$. The ratio of $ze$ and $eg$ together to $zg$ is as the ratio of $eg$ to $gh$. Thus the ratio $ge$ to $gh$ becomes greater than the ratio 571 to 153.

Then the ratio of $eh$ squared to $hg$ squared is as the ratio of $[34]9450$ to $[2]3409$. The ratio of length being greater than the ratio 591$\frac{1}{8}$ to 153. If we add their squares and we take the root of that, $eh$ in this measure is greater than 591$\frac{1}{8}$. We bisect the angle $zeg$ at $h$ by a line $eh$, then the ratio $ze$ to $eg$ is as the ratio $zh$ to $hg$. And $componendo$ and $separando$ the ratio $ze$ and $eg$ combined to $zg$ is as the ratio $eg$ to $gh$. If we add up the numbers that belong to $ze$ [and] $eg$, then that is more than 571. So we make it opposite to $eg$ and what is opposite at $gh$ becomes in this measure 153. If we add their squares and we take the root of that, [then] $eh$ in this measure is greater than 591$\frac{1}{8}$.

Again, let the angle $heg$ be bisected by $eT$, then by the same reasoning we bisect the angle $heg$ at $T$ by a line $eT$. As has been done before; the ratio $ge$ to $gh$ becomes as the ratio 2:1 and let this be as the ratio 306:153. If we subtract the square of the number that is opposite to $zg$ from the square of the number that is opposite to $ez$, and we take the root of what is left, $eg$ is in this magnitude more than 265 and the difference is some fraction [i.e. is less than one].
the ratio $eg$ to $gT$ is greater than the ratio $1162^{1/8}$ to 153, so that the ratio $Te$ to $Tg$ is greater than the ratio $1172^{1/8}$ to 153.

Again, let the angle $Teg$ be divided in halves by the line $ek$. Then the ratio $eg$ to $gk$ is greater than the ratio $2334^{3/8}$ to 153, so that the ratio $gk$ to $hk$ is greater than the ratio $2339^{3/8}$ to 153.

Because the angle $zeg$ was a third of a right angle, it is necessary that angle $leg$ is $1/48$ of a right angle. We make at point $g$ an

Again, let the angle $keg$ be divided in halves by line $le$. Then the ratio $eg$ to $gl$ is greater than the ratio $4673^{3/8}$ to 153.

Because of the angle $zeg$ being a third of a right [angle], the angle $leg$ is $1/48$ of a right [angle]. We construct on the point $g$ of the
angle equal to angle leg, being the angle gem, so that angle lem is $\frac{1}{24}$ of a right angle.
The straight line lm is therefore the side of a polygon containing the circle, having 96 equal angles.
Since we have proved that the ratio eg to gl is greater than the ratio 4673½ to 153, and double of eg is the line ag and double gl is the line lm, it is necessary that the ratio of ag to the perimeter of the polygon with 96 angles is greater than the ratio of 4673½ to 14688, which is greater than its triple by 667½, whose ratio to 4673½ is less than $\frac{1}{7}$.

Thus it is necessary that the polygon containing the circle be greater than triple its diameter by less than $\frac{1}{7}$ of the diameter.

$3\frac{1}{7}$ times the diameter is therefore more than line ge the angle gem equal to the angle gel, then the angle lem is $\frac{1}{24}$ of a right [angle], and the side lm is a side of a regular figure consisting of 96 sides which circumscribe the circle.

Now if we multiply the number that is opposite to lm 96 times, the product of this number reaches 14688. The diameter is in this measure 4673½, so that [number] which is opposite to the perimeter of the figure is more than 3 times that [number] which is opposite to the diameter with an excess of 667½, to which the ratio to the number of the diameter is less than $\frac{1}{7}$. So the perimeter of the [above-] mentioned figure is longer than 3 times the diameter of the circle by less than $\frac{1}{7}$ times the diameter.
And the difference between the circumference of the circle and $3\frac{1}{7}$ times the diameter is greater than that difference necessarily.
the deficit of the circle. Let there be a circle with diameter $ag$ and the angle $bag$ a third of a right angle. Then the ratio of $ab$ to $bg$ is less than the ratio 1351:780, while the ratio of $ag$ to $gb$ is equal to the ratio $1560:780$, because $ag$ is double $gb$.

We repeat the circle. Its diameter is $ag$ and we draw on it the angle $gab$, a third of a right [angle]. Let the ratio $ag$ to $gb$, which is 2:1, be as the ratio 1560:780. Then $ab$ is in this measure less than 1351.

Let angle $bag$ be divided in halves by line $ah$. Since angle $bah$ is equal to angle $hgb$ and angle $bag$ has been divided into halves by line $ah$, it is necessary that angle $hgb$ is equal to angle $hag$; angle $ahg$ is common, so the angles of triangle $ahg$ are equal to the angles of triangle $ghz$.

The ratio of $ah$ to $hg$ is as the ratio of $gh$ to $hz$ and as the ratio of $ag$ to $gz$ and as the ratio of $ah$ to $hg$.

From this it is clear that the ratio of $ah$ to $hg$ is less than the ratio 2911.

We bisect the angle $bag$ by the line $ah$ and we connect $gh$. So in the triangles $ahg$, $ghz$ and $abz$, the angles $hag$, $hgz$ and $baz$ are equal and the two angles [in] $h$ and $b$ are right. [Then] the triangles are similar and because of that the ratio $ah$ to $hg$ is as the ratio $hg$ to $hz$ and as the ratio $ag$ to $gz$ and as the ratio $ah$ to $hz$. Indeed, [it] is as the ratio of $ga$ and $ab$ combined to $gb$. The ratio of $ga$ and $ab$ combined to $gb$ is as the ratio of $ah$ to $hg$. The number of $ag$ and $ab$ combined is less than 2911 and the
Naṣīr al-Dīn al-Ṭūsī’s Version of The ...

to 780, and that the ratio of $ag$ to $gb$ is less than the ratio $301\frac{3}{4}$ to 780.

Let the angle $gah$ be divided in halves by line $aT$, then it is clear from what we said that the ratio of $aT$ to $Tg$ is less than the ratio of $5924\frac{3}{4}$ to 780, which is as the ratio of $1823$ to $240$.

because the ratio of each of the first two numbers to its corresponding one of the other numbers is as the ratio of $3\frac{3}{4}:1$.

Therefore the ratio $ag$ to $gT$ becomes less than the ratio $1838\frac{3}{11}$ to 240.

Further, let the angle $Tag$ be divided in halves by line $ak$. Then the ratio $ak$ to $kg$ is less than the ratio of $3661\frac{9}{11}$ to 240, which is as the ratio of $1007$ to $66$, since the ratio of each of the first two numbers to its corresponding one of the other numbers is as the ratio of $40$ to $11$.

number of $gb$ is 780. So if we add up the two [numbers] which are opposite to $ah$ and $eg$, then $ag$ is in this measure less than $301\frac{3}{4}$.

We bisect the angle $bag$ by the line $aT$ and we connect $Tg$. According to the above-mentioned analogy is [the number] which is opposite to $aT$ less than $5924$ and [the number] opposite to $Tg$ is 780. That is in the ratio $1823:240$.

Let the angle $tag$ be divided in halves by line $ak$, then it is clear from what we said that the ratio of $aT$ to $Tg$ is less than the ratio of $5924\frac{3}{4}$ to 780, which is as the ratio of $1823$ to $240$.

We bisect the angle $tag$ by the line $ak$ and we connect $Tg$. According to the above-mentioned analogy is [the number] which is opposite to $ak$ less than $3661\frac{9}{11}$ and [the number] opposite to $kg$ is 240, that is in the ratio of $1007:66$. That is because the ratio of each of the two [i.e. $3661\frac{9}{11}$ and 240] to
Therefore the ratio of \( \text{ag} \) to \( \text{kg} \) is as the ratio \( 
frac{1009}{6} \) to 66.

Further, let the angle \( \text{kg} \) be divided in halves by line \( \text{la} \). Then the ratio of \( \text{al} \) to \( \text{lg} \) is less than the ratio \( 
frac{2016}{6} \) to 66.

Therefore the ratio of \( \text{ag} \) to \( \text{gl} \) is less than the ratio \( 
frac{2017}{6} \) to 66.

Inverted, the ratio of the perimeter of the polygon, each side being equal to line \( \text{gl} \), to the diameter becomes greater than the ratio of \( 6336 \) to \( 
\frac{2017}{6} \).

We bisect the angle \( \text{lag} \) by the line \( \text{al} \), then [the number] which is opposite to \( \text{al} \) is greater than \( \frac{2016}{6} \) and [the number] opposite to \( \text{lg} \) is 66 and \( \text{ag} \) is in this measure \( \frac{2017}{6} \).

So the ratio \( \text{ag} \) to \( \text{gl} \) is less than the ratio \( \frac{2017}{6} : 66 \). If we multiply 66 by \( \frac{96}{6} \), then all the sides of the polygon of 96 sides inscribed in the circle is \( 6336 \).

But 6336 is greater than 3 times \( \frac{2017}{6} \) by more than \( \frac{10}{71} \).

That is more than 3 times \( \frac{2017}{6} \) with a remainder of \( \frac{10}{71} \) of one [i.e. \( \frac{2017}{6} \) times \( \frac{96}{6} \)].

So the perimeter of the regular figure that is contained in the circle is greater than the addition of the sides of the circle \( 6336 \) to \( \frac{2017}{6} \). That is what we wanted.
I say that there is another method of the astronomers and that is that they obtain a chord of a small arc which is an integer part of the perimeter with the fundamental principles that have been explained in the *Almagest* and other books of them containing demonstrations.

They make it one side of a figure inscribed in the circle. Its ratio to the perpendicular [line] from the center of the circle to it [i.e. the created side] is as the ratio of the side of the similar figure that is circumscribed around the circle to half of the diameter.

So they also deduce this side. Then they obtain, by the computation of these two [quantities] the two quantities, such that the perimeter is greater than one of them and less than the other. So the perimeter is obtained with a close approximation.

The example of that is: let the circle be $ab$, its center is $g$ and $ab$
is $\frac{1}{720}$ part of the perimeter. We connect
the chord $ab$; then its
measure, based on a
calculation of Abū al-
Wafā’ al-Būzjānī
according to the
[above-mentioned
principles with a very
close
approximation, is
$0; 31, 24, 55, 54, 55$. That is a chord of half
a degree, if the
diameter is made 120
parts.
And if we make it [i.e.
the chord] as a side of
a regular figure that
has 720 sides inside
the circle, then the
perimeter of that
figure is according to
his calculation
$376; 59, 10, 59$. If we
bisect the chord of half a degree [then]
the measure of $ad$ is
$0; 15, 42, 27, 57, 27$ and
the square of it is
$0; 4, 6, 44, 2, 4, 57, 25, 18, 30, 9$.
The square of the
radius, that is the line
$ag$, is 3600 parts. We
subtract the square of
$ad$ of it [so that] the
square of $dg$ remains,
which is
$3599; 55, 53, 15, 57, 55, 2, 34, 41, 29, 51$.
The square root of it is
the line $dg$, which is
$59; 59, 57, 56, 37, 56, 51$.  

And if we make it [i.e. the chord] as a side of a regular figure that has 720 sides inside the circle, then the perimeter of that figure is according to his calculation $376; 59, 10, 59$. If we bisect the chord of half a degree [then] the measure of $ad$ is $0; 15, 42, 27, 57, 27$ and the square of it is $0; 4, 6, 44, 2, 4, 57, 25, 18, 30, 9$. The square of the radius, that is the line $ag$, is 3600 parts. We subtract the square of $ad$ of it [so that] the square of $dg$ remains, which is $3599; 55, 53, 15, 57, 55, 2, 34, 41, 29, 51$. The square root of it is the line $dg$, which is $59; 59, 57, 56, 37, 56, 51$. 
We multiply \( ad \) with \( gh \), the radius, and we divide it by \( dg \). The measure of \( he \) results which is \( 0; 15, 42, 28, 29, 45 \).

We double it and it becomes \( 0; 31, 24, 56, 59, 31 \), which is the measure of \( ez \), which is a side of a figure that has 720 sides circumscribed around a circle that is similar to the former circle.

The perimeter of the figure is after calculation \( 376; 16, 59, 23, 54, 12 \).

So if we make the diameter 120, the perimeter [i.e. of the circle] is 376 parts and a fraction that is greater than 0; 59, 10, 59, 0 and less than 0; 59, 23, 54, 12, and if we change the two to the measure that Archimedes mentioned, the perimeter is greater than 3 times the diameter plus something that is greater than \( \frac{10}{70}, \frac{38}{70}, \frac{41}{70}, \frac{21}{70}, \frac{37}{70}, \frac{47}{70}, \frac{37}{70} \), which is approximately \( \frac{10}{70}, \frac{38}{70}, \frac{14}{70}, \frac{29}{70} \).

3) The ratio of the area of every circle to the perimeter of the figure is after calculation 376 as above.

If the perimeter of the circle is \( 3^{1/2} \) times the diameter, then the perimeter is \( 2\pi r \), where \( r \) is the radius. Therefore, the area of the circle is \( \pi r^2 \) times the perimeter, which is \( 2\pi r \cdot \pi r^2 = \pi^3 r^3 \).
Let the line \( ab \) be a diameter of the circle, and let there be made a square \( gh \) around it, and let \( dg \) be \( \frac{1}{7} \) of \( de \). Then, since the ratio of \( ag \) to \( adg \) is as the ratio of 21 to 7, while the ratio of \( agd \) to \( aez \) is as the ratio 7 to 1, it follows that the ratio of triangle \( agz \) to triangle \( adg \) becomes as the ratio of 22 to 7.

But the square \( gh \) is 4 times \( adg \), while the triangle \( ag \) is equal to the circle \( ab \); because the height \( ag \) is equal to this circle’s radius. The base \( gz \) is equal to its circumference, because the circumference of the circle is bigger than 3 times its diameter by about \( \frac{1}{7} \) of it.

What we claimed is thus established: that the ratio of the circle \( ab \) to diameter, and this is an approximate ratio used by the surveyors then the ratio of the surface area of the circle to the square of its diameter is [the] ratio 11 to 14 according to this computation.

So let the diameter of the circle be \( ab \) and draw around it the square \( gh \). Let \( gd \) be one-seventh of \( de \) and \( ez \) be \( \frac{1}{7} \) of \( gd \). Then, since the ratio of \( age \) to \( agd \) is as the ratio of 21 to 7, while the ratio of \( agd \) to \( aez \) is as the ratio 7 to 1, it follows that the ratio of triangle \( agz \) to triangle \( adg \) becomes as the ratio of 22 to 7.

The square of \( gh \) is 4 times \( agd \), and the triangle \( agd \) is equal to the surface area of the circle because \( ag \) is equal to half of the diameter and \( gz \) is approximately equal to the [i.e. the circle’s] perimeter [of the circle].

So the ratio of the square of the diameter to the surface area of the circle is bigger than 3 times its diameter by about \( \frac{1}{7} \) of it.
the square $gh$ is as the ratio of 11 to 14, and that is what we wanted to prove.

the circle is [as] the ratio of 28 to 22, that is, as the ratio of 14 to 11 and that is what we wanted.

This is the completion of the account on the measurement of the circle. Let us finish the treatise by praising God, the most exalted; the good result is because of Him.

1. $H$: مساويا
2. $H$: تساوي
3. $H$: دائرة
4. $H$: اتضح
5. $T$: "ولكن المركز ... الاضلاع عمودا" missing
6. $H$: المتساوي
7. $H$: نصف
8. $H$: مساا
9. $H$: معا
10. $H$: نقطه
11. $H$: الباقي
12. $H$: فالمثلثات
13. $H$: تنصف
14. $T$: لينكن
15. $H$: إذا
16. $H$: مساويا
17. $H$: مساا
18. $H$: النها
19. $H$: واداركبا
20. $H$: adds خط
21. $H$: ١٤٣٨٨
22. $H$: فالسدي
23. $H$: زاوية
24. $H$: اينز
25. $H$: مساوية وزاوية
26. $H$: تكون
27. $H$: $\frac{1}{7}$
28. $H$: جمعها
29. $H$: جمعها
30. $H$: ام $\frac{1}{7}$
31. $H$: جمعها
32. $T$: ٣١٣
33. $H$: جزء
34. $H$: المذكورة التي
35. $H$: فذا
36. $H$: and $T$: هب, scribal error
37. $H$: omitted
38. $H$: omitted
39. $T$: ام
40. $H$: $\overline{\mu}$, $T$: ambiguous
41. $H$: $\overline{\nu}$
42. $T$: possibly $\overline{\nu}$
43. $H$: omitted
44. $H$: omitted
45. $H$: omitted
46. $H$: omitted
47. $H$: مقداره
48. $T$: $\overline{\kappa}$
49. $H$: omitted
50. $T$: omitted
51. $H$: $\overline{\nu}$
52. $H$: $\overline{\xi}$
53. $T$: $\overline{\tau}$
54. $H$: $\overline{\omega}$
55. $T$: possibly $\overline{\nu}$
56. $H$: adds $\overline{\alpha}$

وعشرين بل نسبة اربعة عشر الى احد عشر وذلك ما اردناه

ووهذّا اتمام القول في تكسير الدائرة

ولقطع الكلام حامدين لله تعالى على حسن توفيقه
Figures from Ṭūsī’s text

Figure 1-1: Ṭūsī’s first proposition
Figure 2-1

Figure 2-2

Figure 2: Tūsī’s second proposition, circumscription
Figure 3-1

Figure 3-2

**Figure 3:** Ṭūsī’s second proposition, inscription
Figure 4: Ṭūsī’s second proposition, another proof from the astronomers
The argument of the first proposition

In the first proposition, it is proved that the surface area of a circle is equal to the surface area of a right triangle of which one of the two right-angled sides is equal to the radius and the other the perimeter. Using modern formulas it is easy to see that this is correct: the surface of a triangle is $\frac{1}{2} \times h \times w$ with $h =$ height and $w =$ width. In a right triangle the two right-angled sides are the height and the width and in this special case they are $r$ (the radius) and $2\pi r$ (the perimeter) so the surface area becomes $\frac{1}{2} \times r \times 2\pi r = \pi r^2$ and as we know that is exactly the surface of a circle.

The proof consists of investigating what happens if we assume the circle to be either greater or less than the triangle. In both cases a regular polygon can be constructed such that its surface area is apparently both bigger and smaller, resulting in a contradiction, and so, via reductio ad absurdum, we come to the conclusion that we must
drop the assumption and instead state that the surface area of the circle must be equal to the surface area of the triangle. A brief elaboration on the hypothesis if the circle were bigger should suffice to understand the process; a similar process could be constructed for the hypothesis that the circle is smaller than the triangle. First assume the circle to be greater than the triangle. An inscribed regular polygon is constructed by repeatedly bisecting the chords and connecting the points. See figure 1 and figure 6: if you draw a square inside the circle and outside the circle, the points at which the inside-square touches the circle can be connected with the points at which the outside-square touches the circle, and a regular octagon is constructed. This process can be repeated of course to get a regular 16-sided polygon, 32-sided polygon etc. If the process is repeated, the figure can come indefinitely close to the circle, making sure that it leaves not even an enough room for the circle to be equal to the (supposed) excess of the circle over the triangle (i.e. the regular polygon can be made such that it is bigger than the triangle). That this is the process that should be followed is not explicitly stated by Archimedes who merely states the construction of the inscribed square and the bisection of the arcs. That this process, if repeated, can separate any small amount you like between the circle and the regular polygon is explicitly proved in Euclid’s *Elements* XII, proposition 2. For this case, however, it in principle goes back to the fifth postulate from Archimedes’ *On the sphere and cylinder*, Book I, which states that if there are two quantities of the same kind, then there is a multiple of the difference between these quantities which is greater than the greatest of the two quantities1 (Archimedes, p. 36 and p. 40). This means that a difference cannot be infinitesimal. Where Eutocius’ commentary remains silent on this postulate, Ğūsī does comment on it. He says that this postulate is in turn based on a definition from *Elements* V which states that “magnitudes which have a ratio to one another are those for who it is possible to exceed one another by multiplication” (which is definition 4; see Aṭ-Ḡūsī, pp. 23-24). He then combines this with *Elements* X, proposition 1, concluding that “the smallest of two comparable magnitudes can, by multiplication, become bigger than the biggest of

1. Also known as ‘Archimides’ Postulate’.
Elements X.1, in turn, is a building block for proposition 2 in Elements XII, though this last step is not mentioned by Ţūsī. Nevertheless, Elements XII.2 tells us that any small amount may be separated from a circle by a regular polygon, and since the difference of the circle and the triangle may not be infinitesimal, we know it is possible, by repeatedly bisecting the chords and connection on the points, to make a regular polygon that leaves for the circle less than the difference of the circle and the triangle: the polygon must thus be bigger than the triangle. Ţūsī gives this proof by saying that ‘the resulting triangles separate more than half of the sections’ adding that this has been proved before (it is not clear what he is referring to). Ţūsī also states that this should be repeated until the result is that segments between the circle and the regular polygon are smaller than the excess of the circle over the triangle. The extant Greek text is much more succinct than this, leaving out completely a remark about the repetition of this process. In the early Arabic translation this is touched upon by the comment that “If we have done like that according to what follows...” But it is only with the additions of Ţūsī that the proof becomes entirely clear and unambiguous, when he changes that into “This is repeated until...”.

![Figure 6](image_url)

We now know that the polygon is greater than the triangle. What remains to be proven is that it is simultaneously less than the triangle, arriving at a contradiction. In modern notation this is done by
comparing the formulas of the surface areas of the regular polygon and the triangle. The surface area of the triangle is of course \( \frac{1}{2} \times r \times p \) with \( r \) for radius and \( p \) for perimeter (of the circle). The surface area of the regular polygon is obtained by adding all the surface areas of the triangles that are made (i.e. multiplying the surface area of one triangle by \( n \), for an \( n \)-sided polygon). So this becomes \( n \times \frac{1}{2} \times \alpha \times \beta \) with \( \alpha \) the width of a side and \( \beta \) the height of a triangle (see Figure 6).

From the figure one immediately sees that \( \beta < r \) and \( n \times \alpha < p \). So \( \frac{1}{2} \times \beta \times n \times \alpha < \frac{1}{2} \times r \times p \), so the surface area of the polygon is less than the surface area of the triangle. Ţūsī, in comparison to the Arabic translation, leaves out certain steps in his proof that he probably deemed obvious. First he makes the step from unequal sides to unequal surface areas. He states that the rectangle contained by \( ns \) [the altitude] times the perimeter of the figure is twice the surface area of the regular polygon (which is easily seen to be correct using modern notation) and then states that this is smaller than twice the surface area of the triangle (i.e. the rectangle contained by the radius and the perimeter of the circle). From this he then concludes that the surface area of the regular polygon is smaller than the triangle. The polygon cannot be greater and less at the same time, so we have to drop the assumption that the circle is bigger than the triangle. The proof works likewise if the circle were to be imagined to be less than the triangle. Now the polygon is circumscribed instead of inscribed.

Already in the first paragraph Ţūsī makes an elaboration, where he states that the result is that a circle is equal to a rectangle (sāth) with sides equal to half of the circle’s diameter and half of its perimeter. This seems a comment to make the proposition more precise, and it is this precision that we see come up when comparing the Arabic translation with Ţūsī’s revision. For example, in the second part of the first proposition Archimedes wants to say that after creating a regular polygon around the circle, there will still remain some space left in between the circle and the polygon. The Arabic translation and Ţūsī’s text both say “remaining segments outside the circle”, but the Arabic translation uses the somewhat vague ʿafḍalu ʿalā while Ţūsī uses the unambiguous khārijah min. We may also notice that at other occasions Ţūsī’s text is shorter, though usually this does not decrease
the comprehensibility. For example, the somewhat tedious repetition of ‘and those similar to...’ (wa-naẓāʾ iruhā min...) of the old translation is dropped by Ṭūsī, who offers various alternatives, depending on the context. A rigorous study of the contributions that Ṭūsī made in this respect, especially for the technical vocabulary, will only be possible after we have critical editions of both the early translation(s) of texts like the *Measurement of the circle* and Ṭūsī’s ‘revisions’ of them.

The argument of the second proposition

Ṭūsī’s second proposition is Archimedes’ third proposition. Ṭūsī reorders it probably since in the original the second proposition relies on the result of the third proposition. He follows Archimedes’ reasoning closely and gives no additional information on why the (at first sight) strange numerical ratios are used. In the ratio of \( ez, zg \) and \( eg \) he does however explain the Pythagorean theorem and even states that \( eg \) is 265 and a fraction.

This proposition gives a lower and upper bound for \( \pi \) with a rather sophisticated proof. Before Archimedes, approximations of \( \pi \) were all in a single rational fraction. Here, Archimedes does not give a single rational fraction for \( \pi \), but rather computes a lower and upper bound. As before in the first proposition, understanding one of the two approaches, means also an understanding of the other one as the procedure is the same. It will thus suffice to focus on the upper bound. A regular polygon is constructed circumscribing the circle. The difference with the first proposition is that already at the beginning
valuable information on the size of the constructed side of the regular polygon is taken into account. Since the angle is known (30°) the sides can be calculated. As was said before, we do not know exactly how Archimedes came up with such convenient approximations of $\sqrt{3}$, and likewise we do not know exactly how he made all the following computations. In modern day mathematics it is rather easy to follow. One first starts by constructing a triangle with $A_0=30^\circ$. Then another triangle is formed with $A_{n+1} = \frac{1}{2} \times A_n$. This can be repeated as many times as desired. In the table below all the numbers are given, with the numbers between brackets not given by Archimedes (nor Tūsī). The numbers are with regard to Figure 7 in the order $\text{ab, ac, bc}$. It is noteworthy to see that the numbers of $\text{ab}$ and $\text{ac}$ come closer and closer to each other. In fact, they can come indefinitely close to each other as in reality only points $a$ and $c$ are static (they correspond with points $e$ and $g$ respectively). Point $b$ moves closer and closer to $c$ (see Figure 2).

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If we formalize the procedure we can come up with the following
inequality:
\[ s_n = 6 \times 2^n \times \sin A_n < \pi < S_n = 6 \times 2^n \times \tan A_n \]

Where \( s_n \) is the perimeter of the inscribed polygon divided by the diameter of the circle, \( S_n \) is the perimeter of the circumscribed polygon divided by the diameter of the circle and \( A_n \) is the angle needed to make one side of the polygon. \( A_n \) is then defined as \( A_0 = 30, A_n = \frac{1}{2} A_{n-1} \). This inequality defines a smaller interval as \( n \) increases because the sine and tangent look more and more alike when the angle approaches zero, thus it shows that you can get an estimation of \( \pi \) with an arbitrary small approximation with this method.

In Ţūsī’s text, after the third intersection the numbers go wrong. Instead of 2339\(\frac{1}{4}\) it reads 2339\(\frac{3}{8}\), off by \(\frac{1}{8} \) (thumn). This is then continued through the next numbers. This is not a recalculation by Ţūsī because in the conclusion he returns to the Archimedian value 4673\(\frac{1}{2}\) for the diameter. Maybe this erroneous thumn was added only after Ţūsī but it might as well have been there already before Ţūsī. From the comments Ţūsī makes it is obvious that he did not recalculate them. It could be that the scribe was so used to write a \(\frac{1}{8} \) after a number (as with most of the numbers before) that he just erroneously continued with it. Another mismatch with the numbers happens when he described the inscribing of the circle. After the second intersection he states the value 5924 where Archimedes states 5924\(\frac{3}{4}\). His explanation on why the ratio 5924\(\frac{3}{4}\):780 is equal to 1823:240 is a different approach than Archimedes but is rather obscurely written.

There are not a lot of peculiarities in the text but two are worth mentioning here, both pertaining to this section. The first one is where the translation reads “The diameter is in this measure 4673\(\frac{1}{2}\) ...” The Arabic literally reads “The diameter is in this measure twice 4673\(\frac{1}{2}\) ...”. This ‘twice’ (\(\ddot{d} \tilde{f} \tilde{f} \)) seems out of place and that is also why it is not included in the translation. The other peculiarity is that Ţūsī consistently uses \(\ddot{f} \) (‘in’) to define a polygon inscribing the circle and \(\ddot{c} \ddot{a} \ddot{l} \ddot{a} \) (‘on’) to define a polygon circumscribing the circle. However, where the translation reads “the polygon of 96 sides inscribed in the circle is 6336 ...”, \(\ddot{c} \ddot{a} \ddot{l} \ddot{a} \) is used where \(\ddot{f} \) is expected. Again there is no reasonable explanation and so we can merely neglect it and read it as
if it indicates that the polygon is inscribed in the circle. It is not clear whether we should attribute these slips of the pen to Ṭūsī or to a scribe early in the manuscript tradition.

**The addition to the second proposition**

After the recession of Archimedes’ third proposition, Ṭūsī begins by stating “I say”. This seems to introduce a longer remark, one that is more distant from the original text. That this part of the text does not come from Archimedes may already be clear from the use of sine and cosine functions, nor did he use sexagesimal notation which only after him was introduced in Greek science, coming from Babylonian astronomy. There are other clues as well, already in the first sentence of Ṭūsī’. Here, Knorr translates: “I say, and for the two results <there is> another method,...” (Knorr, p. 581). He translates ‘the two results’ adding in a footnote “literally: “sources”, manjamaini, that is, the two bounds just computed” (Knorr, p. 584). I propose to read munajjimīni instead, being the genitive of the plural of munajjim, meaning, of course, astronomer. Later in the text this is confirmed when Ṭūsī states that this other proof is backed “with the fundamental principles that have been clarified in the Almagest and other certifying books of them.” The Almagest being a book on astronomy, it confirms that Ṭūsī is speaking of a proof that was known among astronomers. It seems that this is a rather crucial correction, as it says something about Ṭūsī’s editorial work and also about the nature of the ‘Middle books’. Saying that this addition relies on astronomers and the Almagest would seem to confirm that indeed most readers were working on the ‘Middle books’ to get to the Almagest and become an astronomer themselves.

The basic idea laid out in this section is to define a chord that is $\frac{1}{720}$th part of an equilateral figure that is inscribed in the circle. All calculations are now done in the sexagesimal system. The whole calculation is based on a number from a trigonometric table of which the chord of $\frac{1}{2}$° is taken. Ṭūsī takes this number from Abu al-Wafā’ Būzjānī, who created a table with trigonometric numbers. From this number, the rest of the construction is calculated via some simple geometrical constructions. First the inscribed polygon is properly
described, then from the numbers of the inscribed polygon, the size of the outer chord is computed (according to Euclid’s *Elements* VI.3). From this number a circumscribed polygon can be constructed and so an upper and lower bound can be established for the circumference of the circle. The conclusion of the proof is similar to the proof of Archimedes, the inner and outer polygons are compared to the diameter of the circle and in addition the mean of the two values is used as a close approximation of \( \pi \).

Unfortunately, Ťūsī is already not correct by assuming the chord of \( \frac{1}{2}^\circ \) to be 0; 31, 24, 55, 54, 55. The correct sexagesimal number is 0; 31, 24, 56, 58, 36, ... and although this is not very far off (by approximately 0.0000046), as Luckey shows the value Ťūsī states is probably mistakenly swapped with the value of \( \sin(\frac{1}{2}^\circ) \), which is close to 0; 31, 24, 55, 54, 0 (Luckey, p. 44). This last value is only \( \frac{55}{60^5} \approx 7 \times 10^{-8} \) off from the value Ťūsī states. Although the error is very small, it could have been better and this would have definitely improved the concluding approximation of \( \pi \). Besides this mistake, the numbers are sometimes hard to read from the manuscript and edition. They can only be confirmed when we calculate ourselves what the numbers should be. For example, the square of \( \text{dg} \) (the square of the apothem, in figure 4) is 3599; 55, 53, 15, 57, 55, 2, 34, 41, 29, 51. However, if we would follow all the possibilities from the footnotes we could also obtain the value 3599; 55, 23, 55, 57, 55, 2, 36, 41, deviating by more than 0.0083. We know the first number is correct as it should add up to 3600 if we add the square of \( \text{ad} \) (using the Pythagorean theorem), so it shows that we cannot trust the text from the manuscript or edition alone.\(^1\) For the apothem of the inscribed polygon, Ťūsī gives 59; 59, 57, 56, 37, 56, 51 which is again really close to the true value of \( 60 \times \cos(\frac{1}{4}^\circ) = 59; 59, 57, 56, 37, 45, ... \) This time off by approximately 0.00000014. Now Ťūsī states that as the side of the inner-polygon stands to the apothem, the side of the outer-polygon stands to the radius. For Ťūsī this makes 0; 31, 24, 56, 59, 31. It it is actually \( 120 \times \tan(\frac{1}{4}^\circ) = 0; 31, 24, 58, 3, ... \) From these values, the

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\(^1\) For example, Knorr gives the erroneous value of 3599; 55, 23, 55, 57, 55, 2, 34, 41. He even gives another erroneous value in a footnote: 3599; 55, 13, 55, 57, 55, 2, 34, 41, 29, 51. It seems that he did not check his numbers. See Knorr, p. 582 and p. 584 note 10.
perimeter must be computed and Ţūsī gives 376; 59, 10, 59. If we would multiply Ţūsī’s calculated side by 720 we would get 376; 59, 10, 58, 59 while the actual number must be 376; 59, 23, 41,… which is off by approximately 0.0036.

A bigger mistake is made in the computation of the perimeter of the outer-polygon, for which Ţūsī gives 376; 16, 59, 23, 54, 12. A first correction on this number is to neglect the 16 which both Luckey and Knorr do. Actually, just a few lines after this value, Ţūsī states it again but this time as 376; 59, 23, 54, 12, correcting his earlier mistake himself. Both Luckey and Knorr do not comment on their emendation. In both the Hyderabad print and the Tehran facsimile it clearly states khāmisah (‘fifth’). This word after the number indicates what the power of sixty is for the last number, so it also indicates how many fractions are to be expected. Also Luckey and Knorr confirm this (Luckey even states that also Woepcke has read this; see Luckey, p. 44). Simply deleting the $\frac{16}{60}$ and stating rābī‘ah, although it is correct and is even used this way by Ţūsī a couple of lines later, does not seem to render the text as it was once written by Ţūsī. In fact, we can show that it was the author’s intention to write khāmisah in this case, instead of assuming a scribal error (or whatever assumption Luckey and Knorr made in silence). For this, we need to do the calculation of the circumference ourselves. We saw that after some calculation Ţūsī gets 0; 31, 24, 56, 59, 31 for one side of a 720-sided polygon that circumscribes the circle. The perimeter of the polygon is then calculated by multiplying this side by 720. So we get 720 × 0; 31, 24, 56, 59, 31 = 12 × 31; 24, 56, 59, 31 (we divide 720 by 60 so we can move the ; one place). We repeat this and obtain:

\[12 \times 31; 24, 56, 59, 31 = \frac{1}{5} \times 31, 24; 56, 59, 31 = 6, 16; 59, 23, 54, 12 = 360 + 16; 59, 23, 54, 12.\]

Here we see what has happened. While 360 + 16 is of course 376 and this would give us the (correct) number 376; 59, 23, 54, 12, Ţūsī must have accidentally copied the 16 from his scrap paper into his manuscript. The khāmisah can only be explained as a quick count of the number of fractions on the side of Ţūsī. As Ţūsī used the correct number just shortly after, it could be argued that Ţūsī was not even consciously aware of his error. This of course does not make it legit to
emendate the text in silence.

One point on the difficulty of these sexagesimal numbers needs to be made before we can move on. This is the number 0; 31, 24, 56, 59, 31 for $\text{hz}$, a side of a 720-sided circumscribing polygon. Knorr thinks it actually reads 0; 31, 24, 57, 59, 31 and elaborates in a footnote that this value, if multiplied by 720 gives a different value (namely, 376; 59, 35, 54, …) from what Ţūsī gives. He then even goes so far as to say that if one would calculate backwards, 376; 59, 23, 54, 12 (the number Ţūsī gives for the perimeter of the circumscribing polygon) would imply a starting value of 0; 31, 24, 56, 59, 31. The difference between six and seven is very ambiguous in manuscripts, since they are represented respectively by $\text{wāw}$ ($\text{ﻭ}$) and $\text{zā}$ ($\text{ﺯ}$) and can thus easily be miscopied, especially given the inconsistent use of diacritics. If evidence shows that one of the two works better than the other, we may assume that it is that value that is intended by the author. Here Knorr recalculate the value to 56 instead of 57 but insists that this is due to a scribal error by the computist and even goes so far as reprimanding Luckey and Woepcke for simply transmitting the values and not checking them (Knorr, pp. 593-594, note 57). It is in fact quite simple to check whether it should be 56 or 57 and that is to reexamine the computation where the number of $\text{hz}$ is based on. Ţūsī computes the number of $\text{hz}$ by doubling the magnitude of $\text{he}$. The magnitude of $\text{he}$ is 0; 15, 42, 28, 29, 45. We can double each individual value and then readjusting the values to the sexagesimal system. Doing this, we get 0; 30, 84, 56, 58, 90. Readjusting gives 0; 31, 24, 56, 59, 30. This gives us a second clue that the value should read 56. First it was noted that when Knorr calculated $\text{hz}$ from the value of the perimeter (calculating backwards) it gave the number 56. Then we saw that if we calculate $\text{hz}$ from $\text{he}$ (calculating forward) it also gave the number 56. It is therefore safe to say that, bearing the ambiguity of the reading of the numbers six and seven in mind, the text really reads 56. On a side note, it is peculiar that the edition and facsimile read 31 as the last number while our computation gave 30. In this case, we can attribute it to a scribal error, but because the two numbers only differ by approximately $1 \times 10^{-9}$ this is not an issue.

Near the end Ţūsī makes from these numbers an approximation that
looks very similar to the approximation of Archimedes. This is in an unusual form of $3^{10/70;38,41,21} \approx 3.141553196379305$, and $3^{10/70;37,47,37} \approx 3.141583110236151$. Ṭūsī gives as an average $3^{10/70;38,14,29} \approx 3.141568151727502$. All these numbers are of the form $3^{x/70}$, with $x$ a fraction. The fraction of the last number is actually the average of the other two fractions (this makes the final approximation not the exact middle value of Ṭūsī’s upper and lower bound). This last value that Ṭūsī states is approximately $2.45 \times 10^{-5} (0.0000245)$ less than the actual value of π. That means that the first four decimals Ṭūsī gives are correct. However, with the combination of a couple of scribal (and possibly calculation) errors and the fact that his results rely on a wrong value of the chord of $\frac{1}{2}^\circ$, we can assume that Ṭūsī was capable of even more accurate approximations of π.

The argument of the third proposition
Here Ṭūsī returns to the original text of Archimedes, following it closely but with one important change. Archimedes opens his proposition by stating that π is equal to $3^{1/7}$ and works the proposition out from this value. Ṭūsī however, merely states that if π is equal to $3^{1/7}$ then the following proposition holds true. This is important as π is actually not really equal to $3^{1/7}$ (as was just concluded in Ṭūsī’s second proposition), rather it is (as Ṭūsī states) a ratio commonly used by surveyors. He also elaborates more on details that Archimedes skips. For example, he does not state the result (that the ratio circle: square is 11:14), but also states the ratio 22:28, which is the ratio actually obtained from the proof.

Concluding remarks
Presenting the evidence through a comparison of Archimedes’ original text and Ṭūsī’s revision, the difference between the two becomes abundantly clear. Already a cursory view of the translations shows that. In the elaboration of the arguments we came across other aspects of Ṭūsī’s text that may not be so obvious from skimming the surface, but which do contribute to the unique character of his revisions and show that these texts are in themselves an original contribution to
mathematics and are therefore not mere imitations of their Greek originals.

Of these aspects, four stand out. First, there is of course the rearrangement of the propositions. This is significant because Ţūsī chose not to make an edition that would be merely imitating Archimedes’ text, but one that was logically correct (in Archimedes’ text, the second proposition logically relies on the third). Secondly, it is obvious that Ţūsī made a conscious effort to streamline the reasoning, adding steps where elucidation was needed and reducing steps where it seemed plain what was meant. Third, and perhaps most important, is of course the highly original addition of Ţūsī to his second proposition. Here, Ţūsī uses a number from a trigonometric table to compute the circumference of a regular polygon that is inscribed in a circle and one that is circumscribed, making two 720-sided polygons. From these two values Ţūsī deduces an estimate of π that is approximately 0.0000245 off of the real value of π. In fact, we noticed that the manuscripts contain scribal errors so Ţūsī’s real computed value could have been an even closer estimation. Fourth, Ţūsī seemed to sense the error in the statement that π is equal to 3½ (this is asserted in Archimedes’ third proposition and is needed to complete the proof of the proposition) and he corrected this into an if clause, remarking that this is an approximation regularly used by surveyors.

If the small treatise on the measurement of the circle is any indication of the nature of all of the Tahīr al-mutawassiṭāt, ‘Revision of the middle books’, then we can conclude that these revisions almost function as a commentary to their original versions, indeed, exceeding earlier commentaries. They give evidence to Ţūsī’s mathematical acumen, and further give a unique insight in the state of knowledge of Ţūsī’s time, not only of mathematics but also for example of astronomy and land surveying. Given the impact these treatises had in the centuries after Ţūsī, we can only hope that the study on these important documents will be continued in the future.
References
